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HW 2

MATH 534A
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1. Tu, 5.1

Let $A$ and $B$ be two points not on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} \setminus \{0\}) \cup \{A, B\}$.

For any two positive real numbers $c, d$, define

$$I_A(-c, d) = (-c, 0) \cup \{A\} \cup (0, d)$$

and similarly for $I_B(-c, d)$, with $B$ instead of $A$. Define a topology on $S$ as follows: On $(\mathbb{R} \setminus \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis at $A$ is the set $\{I_A(-c, d) | c, d > 0\}$; similarly, a basis at $B$ is $\{I_B(-c, d) | c, d > 0\}$.

(a) Prove that the map $h : I_A(-c, d) \to (-c, d)$ defined by

$$\begin{aligned} h(x) &= x \text{ for } x \in (-c, 0) \cup (0, d) \\ h(A) &= 0, \end{aligned}$$

is a homeomorphism.

**Solution.** First, a useful definition,

**Definition.** A function $f : X \to Y$ between two topological spaces $(X, T_X)$ and $(Y, T_Y)$ is a *homeomorphism* if

- $f$ is a bijection,
- $f$ is continuous,
- $f^{-1}$ is continuous.

Clearly, $h$ is a bijection, since $h^{-1}$ is a function, where $h^{-1}$ is defined as follows

$$h^{-1}(y) = \begin{cases} y & y \in (-c, 0) \cup (0, d) \\ A & y = 0 \end{cases}$$

Now, is $h$ continuous? Yes, because the limit from the right or left of $h$ at 0 is 0, precisely what $h(A)$ is defined to be. Finally, is $h^{-1}$ continuous? Check this via the definition: Is the pullback of an open set from $(-c, d)$ under $h^{-1}$ open in the topology generated on $I_A(-c, d)$? Consider the set $(-c, d)$ itself. This set is open in the topology on $\mathbb{R}$ with pullback $I_A(-c, d)$. Because we are given that the set of all such $I_A$ forms a basis at $A$, we can conclude that $I_A(-c, d)$ is open and $h^{-1}$ is a continuous map at $A$. Continuity at all other points follows immediately from the definition of $h^{-1}$. □

(b) Show that $S$ is locally Euclidean and second countable, but not Hausdorff.

**Solution.** Locally Euclidean follows from above; since we have shown that at each point $p \in S$ there is a neighborhood $U$ where a homeomorphism exists taking $U$ to an open subset of $\mathbb{R}$. Here, we need to utilize the construction of bases at $A$ and $B$ in (a).

Second countable is easy as well; we just need to construct a countable basis for $S$. Here it is

$$\begin{aligned} B_n &= \begin{cases} (-1/n, 0) \cup \{A\} \cup (0, 1/n) & n \text{ even} \\ (-1/n, 0) \cup \{B\} \cup (0, 1/n) & n \text{ odd} \end{cases} \\ \mathcal{B} &= \{B_n | n \in \mathbb{N}\} \end{aligned}$$

Now, I want to show that it fails to be Hausdorff. Assume that $S$ is Hausdorff. This would mean that I can find open sets in the topology on $S$ s.t. for $A \in U$ and $B \in V$, $U \cap V = \emptyset$. However, from the definition of the bases at $A$ and $B$, such a construction is not possible, as there will be overlap of the open intervals defined around $A, B$. Therefore the intersection of $U$ and $V$ will be non-empty. Hence, $S$ cannot be Hausdorff. □

2. Tu, 6.6

   Show that a map $f : M \to N$ of manifolds is $C^\infty$ if and only if for every chart $(U, \phi)$ in the atlas of $M$ and $(V, \psi)$ in the atlas of $N$, the composite $\psi \circ f \circ \phi^{-1}$ is $C^\infty$ on $\phi(f^{-1}(V) \cap U)$.

   **Solution.** ($\Longleftarrow$) Assume that for every chart $(U, \phi)$ in the atlas of $M$ and $(V, \psi)$ in the atlas of $N$, the composite $\psi \circ f \circ \phi^{-1}$ is $C^\infty$ on $\phi(f^{-1}(V) \cap U)$. The pullback of $V$ into $M$ clearly contains $p$ and therefore, by Definition 6.3, $f$ is $C^\infty$ at $p$. This argument can be extended to all points $q$ in the intersection $f^{-1}(V) \cap U$ for all open sets $U \in M$ and $V \in N$. The collection of all such open sets comprises the spaces $M$ and $N$. Therefore, $f : M \to N$ is a $C^\infty$ map.

   ($\Longrightarrow$) Assume that a map $f : M \to N$ of manifolds is $C^\infty$. This means that for each point $p$ in $M$ there are charts $(U, \phi)$ containing $p \in M$ and $(V, \psi)$ containing $f(p) \in N$ s.t. the map $\psi \circ f \phi^{-1}$ is $C^\infty$ at $p$. By continuity of $f$, a set $U$ can always be chosen s.t. $f(U) \subset V$. This ensures that the composite map $\psi \circ f \circ \phi^{-1}$ will be $C^\infty$ at all points $q$ in the intersection of $f^{-1}(V) \cap U = U$. Because $f$ is $C^\infty$ for all points $p \in M$, the argument extends to all charts at all points $q \in M$ and $f(q) \in N$. Therefore, every chart $(U, \phi)$ in the atlas of $M$ and $(V, \psi)$ in the atlas of $N$, the composite $\psi \circ f \circ \phi^{-1}$ is $C^\infty$ on $\phi(f^{-1}(V) \cap U)$. □

3. Tu, 6.7

   Show that the general linear group $\mathrm{GL}(n, \mathbb{R})$ defined in Example 5.14 is a Lie group under matrix multiplication.

   **Solution.** Clearly, the product of matrices with non-zero determinant is itself a matrix with non-zero determinant. So, $\mathrm{GL}(n, \mathbb{R})$ is closed under matrix multiplication.

Component-wise, matrix multiplication is defined by, for elements $A, B \in \mathrm{GL}(n, \mathbb{R})$,

$$(AB)_{ij} \quad = \quad \sum_{k=1}^{n} a_{ik} b_{kj}.$$

This is a polynomial in the coordinates of both $A, B$, therefore matrix multiplication is $C^\infty$.

Now, it remains to show that the inverse map is $C^\infty$. From Cramer's rule, we know that

$$(A^{-1})_{ij} = \frac{1}{\det A}(-1)^{j+i}\big((j,i) - \text{minor of } A\big),$$

where the $(j,i)$-minor of a matrix is defined as the determinant of the submatrix obtained from the deletion of the $j^{th}$ row and the $i^{th}$ column of $A$. This map is $C^\infty$ as long as the $\det A \neq 0$, but this is guaranteed in $\mathrm{GL}(n, \mathbb{R})$.

Therefore, $\mathrm{GL}(n, \mathbb{R})$ is closed under matrix multiplication with both the multiplication and inverse maps $C^\infty$. Therefore $\mathrm{GL}(n, \mathbb{R})$ is a Lie group. □

4. Tu, 6.8

Let $V$ be a finite-dimensional vector space over $\mathbb{R}$, and $\mathrm{GL}(V)$ the group of all linear isomorphisms of $V$ itself. A basis $e_1, \ldots, e_n$ for $V$ induces a bijection

$$\begin{aligned} \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{GL}(V), \\ [a^i_j] &\mapsto (e_j \mapsto \sum_i a^i_j e_i), \end{aligned}$$

making $\mathrm{GL}(V)$ into a $C^\infty$ manifold, which we denote temporarily by $\mathrm{GL}(V)_e$. If $\mathrm{GL}(V)_u$ is the manifold structure induced from another basis $u_1, \ldots, u_n$ for $V$, show that $\mathrm{GL}(V)_e$ is diffeomorphic to $\mathrm{GL}(V)_u$.

**Solution.** To show that the two smooth manifolds are diffeomorphic, it is natural to try to find a suitable diffeomorphism. In this instance, the most natural such diffeomorphism would be the identity map. That is, take a vector $v = \sum_j c_j e_j \in \mathrm{GL}(V)_e$ and map it to itself in $\mathrm{GL}(V)_u$. Clearly, the identity map is a diffeomorphism, but is this map reasonable here?

Since we are considering the action of all linear operations (with non-zero determinant) on a vector space $V$, this is a suitable map from $\mathrm{GL}(V)_e$ to $\mathrm{GL}(V)_u$; i.e. there will always be a linear transformation that, when the change of basis formula has been taken into account, will create the same vector in $\mathrm{GL}(V)_u$ that is in $\mathrm{GL}(V)_e$. In this case, the choice of basis provides us with the capability of putting a $C^\infty$ manifold structure on a vector space. However, the specific basis that we use is not important; it is only important that we have a basis. Therefore, for any choice of basis, the identity map will always provide a diffeomorphism.

Therefore, $\mathrm{GL}(V)_e$ is diffeomorphic to $\mathrm{GL}(V)_u$. □

5. Tu, 7.1

   Suppose a left action of a topological group $G$ on a topological space $S$ is continuous; this simply means that the map $G \times S \to S$ describing the action is continuous. Define two points of $x, y$ of $S$ to be equivalent if there is a $g \in G$ such that $y = gx$. Let $G \setminus S$ be the quotient space. Prove that the projection map $\pi : S \to G \setminus S$ is an open map.

   **Solution.** To show that the projection map $\pi$ is open, I need to show that $\pi^{-1}(\pi(U))$ is open in $S$. Take an arbitrary open set $U \in S$,

$$\begin{aligned} \pi^{-1}\big(\pi(U)\big) &= \bigcup_{y \in U} [y] \\ &= \bigcup_{g \in G} gU. \end{aligned}$$

   For any $g \in G$, multiplication by $g$ creates a homeomorphism on $S$. Therefore, the set $gU$ is open in $S$. Since a topological space is closed under arbitrary union, $\cup_{g \in G} gU$ is open. Therefore, $\pi$ is an open map. □

6. Tu, 7.3

   Show that the real projective space $\mathbb{R}P^n$ is compact. (*Hint*: Use Exercise 7.11).

   **Solution.** If I can show that $\mathbb{R}P^n$ is homeomorphic to something which is compact, then it would follow that $\mathbb{R}P^n$ is compact.

   The hint given is to use Exercise 7.11, which provides a map that would induce a homeomorphism on $S^n / \sim$, which is compact. So, to finish this problem, I need to complete Exercise 7.11.

   *7.11* Prove that the map $f : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ given by

$$f(x) = \frac{x}{|x|}$$

   induces a homeomorphism $\bar{f} : \mathbb{R}P^n \to S^n / \sim$. (*Hint*: Find an inverse map

$$\bar{g} : S^n / \sim \longrightarrow \mathbb{R}P^n$$

   and show that both $\bar{f}$ and $\bar{g}$ are continuous.)

   **Proof.** Define $\bar{f} : \mathbb{R}P^n \to S^n / \sim$ by $\bar{f}([x]) = \left[\frac{x}{|x|}\right] \in S^n / \sim$. This map is well defined because $\bar{f}([tx]) = \left[\frac{tx}{|tx|}\right] = \left[\frac{x}{|x|}\right]$ are the projection maps to the quotient spaces. By Proposition 7.1, $\bar{f}$ is continuous because $\pi_2 \circ f$ is continuous.

   Now define $g : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ by $g(x) = x$. This map induces another map $\bar{g} : S^n / \sim \longrightarrow \mathbb{R}P^n$, $\bar{g}([x]) = [x]$. From above, $\bar{g}$ is well defined and continuous. Additionally,

$$\begin{aligned} \bar{g} \circ \bar{f}([x]) &= \left[\frac{x}{|x|}\right] = [x] \\ \bar{f} \circ \bar{g}([x]) &= [x], \end{aligned}$$

   So, $\bar{f}$ is continuous with continuous inverse $\bar{f}^{-1} = \bar{g}$. Therefore, $\bar{f}$ is a homeomorphism from $\mathbb{R}P^n$ to $S^n / \sim$.

Therefore, $\mathbb{R}P^n$ is homeomorphic to the compact $S^n/\sim$ and $\mathbb{R}P^n$ is hence compact. $\square$

7. Use the Inverse Function Theorem to do Problem 1.4 with $F(x)$ replaced by $x + F(x)$.

   **Solution.** Define the map $F : B(0, \pi/2) \subset \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x) = \begin{cases} x + h(|x|)\frac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

   We know that $h$ is a diffeomorphism on $\mathbb{R}^+\backslash\{0\}$, therefore we don't need to worry about differentiability of $F, F^{-1}$ due to $h, h^{-1}$. The problem is that $F^{-1}$, as was previously defined in the book is not differentiable at 0. If we can add differentiability at 0 for $F^{-1}$, we are done. However, adding $x$ we can use the Implicit Function Theorem to show that $F^{-1}$ is differentiable at 0. If I can show that the determinant of the Jacobian of $F$ is nonzero at 0, then $F$ will be locally invertible at 0, meaning $F^{-1}$ is $C^\infty$ at 0. The determinant of the Jacobian is $\prod_{i=1}^n \partial F/\partial x^i|_{x=0} = \prod_{i=1}^n (1 + (h(|x|)x/|x|)_i)|_{x=0}$, which will be nonzero since all partial derivatives $(h(|x|)x/|x|)_i)|_{x=0}$ will be positive semi-definite.

   Therefore, $F$ is a diffeomorphism. $\square$